Derivative of the Exponential Map

We want to find the derivative of the rotation matrix $R(v) = \exp(v_\times)$ with respect to the rotation vector v. The result will be a 3rd order tensor. To avoid dealing with tensor notation, we will look at the effect the rotation has on an arbitrary vector X: $F(v) = R(v)X$. The function F is a function from a vector space to a vector space, and hence its derivative is a 3×3 matrix.

To calculate $\mathrm{D}_v F$ we use a trick where we replace a function by the integral of its derivative with respect to an auxiliary variable s:

$$\begin{aligned}\mathrm{D}_v F &= \exp(v_\times)\left(\exp(-v_\times)\,\mathrm{D}_v \exp(v_\times)X\right)\\ &= \exp(v_\times)\int_0^1 \frac{d}{ds}\left[\exp(-sv_\times)\,\mathrm{D}_v \exp(sv_\times)X\right] ds\end{aligned}$$

we apply the product rule to the derivative with respect to s:

$$= \exp(v_\times)\int_0^1 \left[-v_\times \exp(-sv_\times)\,\mathrm{D}_v \exp(sv_\times)X + \exp(-sv_\times)\,\mathrm{D}_v\left(v_\times \exp(sv_\times)X\right)\right] ds$$

and the product rule to the derivative with respect to v:

$$= \exp(v_\times)\int_0^1 \exp(-sv_\times)\left(\mathrm{D}_v\, v_\times\right)\exp(sv_\times)X\,ds$$

At this point we need to calculate the derivative of $v_\times$: for any vector Z, we have

$$(\mathrm{D}_v\, v_\times)Z = \mathrm{D}_v(v \times Z) = -\,\mathrm{D}_v(Z_\times v) = Z_\times\,\mathrm{D}_v\, v = -Z_\times$$

Applying this to to the derivative of F:

$$\begin{aligned}\mathrm{D}_v F &= -\exp(v_\times)\int_0^1 \exp(-sv_\times)\left(\exp(sv_\times)X\right)_\times ds\\ &= -\int_0^1 \exp((1-s)v_\times)\left(\exp(sv_\times)X\right)_\times ds\end{aligned}$$

For any rotation matrix R, we have $RZ_\times = (RZ)_\times R$, and so:

$$\begin{aligned}&= -\left(\exp(v_\times)X\right)_\times \int_0^1 \exp((1-s)v_\times)ds\\ &= -\left(R(v)X\right)_\times \int_0^1 R(tv)dt\\ &= -\left(R(v)X\right)_\times T(v)\end{aligned}$$

The integral $T(v) = \int_0^1 \exp(sv_\times)ds$ can be computed using the Rodrigues' formula:

$$\begin{aligned}T(v) &= \int_0^1 \exp(sv_\times)ds\\ &= \int_0^1 \left(\mathrm{Id} + \frac{\sin(sa)}{sa} sv_\times + \frac{1-\cos(sa)}{(sa)^2}(sv_\times)^2\right) ds\\ &= \int_0^1 \left(\mathrm{Id} + \frac{\sin(sa)}{a} v_\times + \frac{1-\cos(sa)}{a^2} v_\times^2\right) ds\\ &= \mathrm{Id} + \frac{1-\cos(a)}{a^2} v_\times + \frac{1}{a^2}\left(1 - \frac{\sin(a)}{a}\right) v_\times^2\end{aligned}$$

Properties

1. Since the rotation vector v is invariant under $R(v)$, it is also invariant under $T(v)$:

$$T(v)v = \int_0^1 R(sv)v ds = \int_0^v ds = v$$

2. Let's prove the property of reflectivity:

$$\begin{aligned} R^t(v)T(v) &= \int_0^1 R(-v)R(sv)ds \\ &= \int_0^1 R((s-1)v)ds \ \ \text{(change variables } t = 1-s\text{)} \\ &= \int_0^1 R(-tv)dt \\ &= \int_0^1 R^t(tv)dt \\ &= T^t(v) \end{aligned}$$

3. And the double angle formula:

$$\begin{aligned} T(2v) &= \int_0^1 R(2sv)ds \ \ \text{(change variables } t = 2s\text{)} \\ &= \int_0^2 R(tv)dt \\ &= \int_0^1 R(tv)dt + \int_1^2 R(tv)dt \ \ \text{(change variables } p = t-1\text{)} \\ &= T(v) + \int_0^1 R((p+1)v)dp \\ &= T(v) + T(v)R(v) \\ &= T(v)(\mathrm{Id} + R(v)) \end{aligned}$$